Pricing Double Barrier Options Using Reflection Principle

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Abstract

The pricing of double barrier option contracts has motivated number of academic studies in the recent years. This paper provides a simpler and more efficient close-form pricing formula with the application of twice reflection principle to deviate from the existing models that yield an analytical or numerical solution solely. Behind the main formula, solutions of double barrier-hitting joint probability are also derived. In verifying the validity of our method, the results of numerical comparisons indicate that the values computed by our formula fit those from Monte Carlo simulation and Laplace-transform method very well. The sizes of error ratios are slight and stably ranged from –0.1% to 0.1%.

Keywords: Double Barrier Option, Reflection Principle, Option Pricing, Barrier  
JEL Classification Codes: G13

1. Introduction

Since the well-known Black-Scholes formula was proposed in 1973, option pricing has already received a lot of attention from academic researches. Among the exotic options, “Barrier option” plays one of the most popular types of path-dependent options, due to the fact that it is cheaper than standard option but offer a similar kind of protection. Such a type of option, as implied by its name, is an option contract whose payoffs depend on the relationship between the predetermined barriers and the path of prices of the underlying asset. If the barrier serves as a knock-out mechanism, the option becomes nullified instantly once the underlying asset price touches the barrier during the trading period. Reversely, the option with knock-in barrier only comes in existence if the asset price crosses the in-barrier.
A straight extension to “single barrier” options is to consider “double barrier” options characterized by double terms of the contracts. Double barrier options are an ideal method for investors in particular to take advantage of target markets to trade within a range. Most common in the foreign exchange, they are available in most underlying markets, including commodities, interest rate and equity markets. Under different methods, prior works that have already devoted to double barrier options pricing includes Kunitomo and Ikeda (1992), Geman and Yor (1996), Baldi et al. (1999), Pelsser (2000), Schroder (2000), Guillaume (2003), and Lo and Hui (2007).

Based on the Levy formula, Kunitomo and Ikeda (1992) derive the probability density for standard Brownian motion staying between two (exponentially) curved boundaries. They express the density as an infinite sum of normal density functions, and the values of double knock-out options are derived by integrating with respect to this density. Geman and Yor (1996) and Pelsser (2000) invert the Laplace transform to derive the analytical solutions of double barrier options. In 1999, the work by Baldi et al. develops “Sharp Large Deviation” method for improving Monte Carlo simulation techniques in pricing the double barrier options. Schroder (2000) deals with this pricing problem using the Cauchy Residue Theorem. The method of Guillaume (2003) is mainly based on the Markov Property of Brownian motion. More recently, Lo and Hui (2007) make the application of Fourier series expansion in valuing the double barrier option with time-dependent parameters.

Most of the above existing literatures, however, merely provide the analytical solutions for double barrier option prices, rather than a simple and exact closed-form formula. Further, they usually deal with only one type of double barrier option: double barrier knock-out options. In the markets a much wider variety of double barrier options is being traded (e.g., double knock-in options). Thus, the motivation for our research is to overcome this problem.

Differing from the prior studies, we apply the martingale method and twice reflection principle of geometric Brownian motion in pricing the double barrier option. The application of reflection principle is really wide, and contains several aspects, such as physics, engineering, financial modeling, etc. This also motivates a large numbers of studies in many kinds of topics. The doing of this paper plays an interesting extension of the technique by Liao and Wang (2002). Our contribution is twofold:

First, the formulas of whole types of double-barrier-hitting joint probabilities are derived, including double knock-out, double knock-in, up-out and down-in, and up-in and down-out. Functions we provide are close-form and expressed as the combination of bivariate normal distribution. These are very useful in deriving the value of finance derivatives with path-dependent payoffs.

Second, we give more simpler and explicit solutions (in close-form) for the value of whole types of double barrier option. These can be served as the building blocks of put-call and in-out parities for barrier options. Through the numerical investigation and analysis, the result indicates that the values computed by our formula fit those from Monte Carlo simulation and by Pelsser’s Laplace-transform approach very well.

The rest of this paper is organized as follows. In Section 2 below we develop the pricing model for double barrier option with making the application of martingale method and twice reflection principle. Section 3 numerically presents the comparison between the approach from existing literature and ours. The conclusion will be drawn in Section 4.

2. The Pricing of Double Barrier Option
2.1. Preliminaries

Fix a probability space \((\Omega, \mathcal{F}, Q)\), endowed with the filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) that is rich enough to support whole stochastic process in the continuous-trading economy. Assume that the markets are perfect and frictionless; namely, there are no transaction costs and tax.

Now consider in this derivative market a tradable asset with current value \(S(0)\), which motivates investors to write the double barrier option contract with strike price \(K\), upper barrier \(H\),
lower barrier $L$, and monitored period $[0, \tau]$. The dynamic of its price is formed by the following stochastic differential equation:

$$dS(t) / S(t) = (r_d - \delta) dt + \sigma_S \cdot dW(t)$$

where the constant parameters $r_d$, $\delta$, $\sigma_S$ stand for the domestic interest rate, the dividend-paying ratio, and the positive volatility respectively; $W(t)$ plays a single-dimensional Brownian motion with respect to $\mathbb{F}$ under the risk-neutral measure $Q$. To derive the formulas of double-barrier-hitting joint probability, we shall make the introduction to an interesting property of this Brownian motion – reflection principle.

**Lemma 1.** Let the stochastic process $\{Y(t)\}_{t \geq 0}$ where $t \in \mathcal{R}_+$ follows a Brownian motion on $(\Omega, \mathcal{F})$ with stopping time $\tau_a := \inf\{t \in \mathcal{R}_+: Y(t) = a\}$. Define

$$\tilde{Y}(t) = \begin{cases} Y(t) & \text{if } \tau_a > t \\ 2a - Y(t) & \text{if } \tau_a \leq t \end{cases}$$

then the process $\{\tilde{Y}(t)\}_{t \geq 0}$ is also a Brownian motion that preserves the distribution same with the original one.

Simply speaking, the reflection principle tells us that reflected Brownian motion after hitting a barrier preserves the same motion as the original one. Using this Lemma twice, the path-dependent characters of underlying asset price can be examined.

**Theorem 2.** Let $X(t) = \nu t + \sigma_S W(t)$, $x = \ln[H / S(0)] \geq 0$, $y = \ln[L / S(0)] \leq 0$ where $\nu = r_d - \delta - 0.5\sigma_S^2$. Denote by $M^X_\tau = \sup_{0 \leq t \leq \tau} X(t)$ and $m^X_\tau = \inf_{0 \leq t \leq \tau} X(t)$, the maximum and minimum of sample-path of underlying asset log-price during the period $[0, \tau]$ respectively. Given the current information set $\mathcal{F}_0$, the joint probabilities of $M^X_\tau$ and $m^X_\tau$ are

$$Q(\omega: M^X_\tau(\omega) \geq x \text{ or } m^X_\tau(\omega) \leq y) = N([-x + \nu \tau] / (\sigma_S \sqrt{\tau})) + N([-y + \nu \tau] / (\sigma_S \sqrt{\tau})) + \exp(2\nu \sqrt{\tau} / \sigma_S^2) \times \{N([-y + \nu \tau] / (\sigma_S \sqrt{\tau})) - N([-y - x + \nu \tau] / (\sigma_S \sqrt{\tau}))\} \times \exp(2\nu x / \sigma_S^2) - \{N([-y - x + \nu \tau] / (\sigma_S \sqrt{\tau})) \exp(2\nu (y - x) / \sigma_S^2);$$

$$Q(\omega: M^X_\tau(\omega) \leq x \text{ and } m^X_\tau(\omega) \geq y) = N([-x - y + \nu \tau] / (\sigma_S \sqrt{\tau})) - N([-y - x + \nu \tau] / (\sigma_S \sqrt{\tau})) + \exp(2\nu \sqrt{\tau} / \sigma_S^2) \times \{N([y - x - \nu \tau] / (\sigma_S \sqrt{\tau})) - N([y - x + \nu \tau] / (\sigma_S \sqrt{\tau}))\} \times \exp(2\nu x / \sigma_S^2) + \{N([-y - x - \nu \tau] / (\sigma_S \sqrt{\tau})) \exp(2\nu (x - y) / \sigma_S^2);$$

$$Q(\omega: M^X_\tau(\omega) \geq x \text{ and } m^X_\tau(\omega) \geq y) = N([-x - y + \nu \tau] / (\sigma_S \sqrt{\tau})) - \exp(2\nu \sqrt{\tau} / \sigma_S^2) N([y - x - \nu \tau] / (\sigma_S \sqrt{\tau})) \times \exp(2\nu x / \sigma_S^2) - \{N([-y - x - \nu \tau] / (\sigma_S \sqrt{\tau})) \exp(2\nu (y - x) / \sigma_S^2);$$

$$\ldots$$
Q(ω:M^x(ω) ≤ x and m^y(ω) ≤ y) = \exp (2νy/σ_2^2)N[(y−2x+ντ)/σ_2√τ] \exp (2νy/σ_2^2)N[(y−2x−ντ)/σ_2√τ] 

where N(· ) refers as a cumulative standard normal distribution.

Formulas given in Theorem 2 provide the critical clues to solve the in-the-money probability for double barrier options. Further application on the valuation would be discussed in next subsection. For the detailed proof of this theorem, see Appendix.

2.2. Pricing Model for Double Barrier option and Close-form Solutions

Consider now a double-barrier-option market that allows for continuous trading and contains eight types of the zero-rebate European contract; namely, double knock-in call (C_{DKI}), double knock-out call (C_{DKO}), up-out and down-in call (C_{UODI}), up-in and down-out call (C_{UIDO}), double knock-in put (P_{DKI}), double knock-out put (P_{DKO}), up-out and down-in put (P_{UODI}), and up-in and down-out put (C_{UIDO}). The structures of contractual payoff at the termination (time-T) are displayed as below:

C_{DKI} (T) \equiv [S(T) − K]^+ 1_{\{M^x ≥ H \cap m^y ≤ L\}} 
= [S(T) − K] 1_{\{S(T) > K\}} (1_{\{M^x ≥ H\}} + 1_{\{m^y ≤ L\}} - 1_{\{M^x ≤ H\}} 1_{\{m^y ≤ L\}})

C_{DKO} (T) \equiv [S(T) − K]^+ 1_{\{M^x ≤ H \cap m^y ≤ L\}} 
= [S(T) − K] 1_{\{S(T) > K\}} 1_{\{M^x ≤ H\}} 1_{\{m^y ≤ L\}}

C_{UODI} (T) \equiv [S(T) − K]^+ 1_{\{M^x ≥ H \cap m^y ≤ L\}} 
= [S(T) − K] 1_{\{S(T) > K\}} 1_{\{M^x ≥ H\}} 1_{\{m^y ≤ L\}}

C_{UIDO} (T) \equiv [S(T) − K]^+ 1_{\{M^x ≥ H \cap m^y ≤ L\}} 
= [S(T) − K] 1_{\{S(T) > K\}} 1_{\{m^y ≥ L\}} 1_{\{m^y ≤ L\}}

P_{DKI} (T) \equiv [K − S(T)]^+ 1_{\{M^x ≥ H \cap m^y ≤ L\}} 
= [K − S(T)] 1_{\{S(T) < K\}} (1_{\{M^x ≥ H\}} + 1_{\{m^y ≤ L\}} - 1_{\{M^x ≤ H\}} 1_{\{m^y ≤ L\}})

P_{DKO} (T) \equiv [K − S(T)]^+ 1_{\{M^x ≤ H \cap m^y ≥ L\}} 
= [K − S(T)] 1_{\{S(T) < K\}} 1_{\{M^x ≤ H\}} 1_{\{m^y ≥ L\}}

P_{UODI} (T) \equiv [K − S(T)]^+ 1_{\{M^x ≥ H \cap m^y ≥ L\}} 
= [K − S(T)] 1_{\{S(T) < K\}} 1_{\{M^x ≥ H\}} 1_{\{m^y ≥ L\}}

As the doing in earlier works, this paper starts from pricing the double knock-in call. Based on the martingale property, a double knock-in call contract has the current value

C_{DKI} (0) = B_d(0) \mathbb{E}_Q [C_{DKI} (T) / B_d (T) | F_0] \tag{1}

where B_d represents the saving account satisfying dB_d(t) = r_d B_d(t) dt and B_d(0) = 1. Substituting the contingent payoff into (1) yields

C_{DKI} (0) = B_d(0) \mathbb{E}_Q \left[ [S(T) − K] 1_{\{S(T) > K\}} (1_{\{M^x ≥ H\}} + 1_{\{m^y ≤ L\}} - 1_{\{M^x ≤ H\}} 1_{\{m^y ≤ L\}}) / B_d (T) | F_0 \right]. \tag{2}
To solve the conditional expectations in (2), formulas of in-the-money probability for double barrier options are required. These will be given in the following theorem.

**Theorem 3.** Given the current information set $F_0$, for any $0 < \tau \leq T$ and $x \geq z \geq y$, formulas of the in-the-money probability for double barrier calls are

$$Q\{\omega : M^y_T (\omega) \geq x \text{ or } m^x_T (\omega) \leq y\} \cap \{\omega : X(\omega, T) \geq z\}$$

$$= N_z\left(\frac{(-z + v T)}{\sigma_x \sqrt{T}}, \frac{(-x + v T)}{\sigma_y \sqrt{T}}, \rho, \right)$$

$$+ N_z\left(\frac{(-z + v T)}{\sigma_x \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$+ \exp\left(2 v x / \sigma_x^2\right) N_z\left(\frac{2 x - z + v T}{\sigma_x \sqrt{T}}, \frac{(-x + v T)}{\sigma_y \sqrt{T}}, \rho, \right)$$

$$- N_z\left(\frac{2 x - z + v T}{\sigma_x \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$+ \exp\left(2 v y / \sigma_y^2\right) N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-x + v T)}{\sigma_x \sqrt{T}}, \rho, \right)$$

$$- N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-z + v T)}{\sigma_x \sqrt{T}}, \rho, \right)$$

$$+ \exp\left(2 v y / \sigma_z^2\right) N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$- N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-z + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$+ \exp\left(2 v (y-x) / \sigma_x^2\right) N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$- N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$= N_z\left(\frac{(-z + v T)}{\sigma_x \sqrt{T}}, \frac{(-y + v T)}{\sigma_y \sqrt{T}}, \rho, \right)$$

$$- \exp\left(2 v y / \sigma_y^2\right) N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$+ \exp\left(2 v x / \sigma_x^2\right) N_z\left(\frac{2 x - z + v T}{\sigma_x \sqrt{T}}, \frac{(-x + v T)}{\sigma_y \sqrt{T}}, \rho, \right)$$

$$- N_z\left(\frac{2 x - z + v T}{\sigma_x \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$- \exp\left(2 v (y-x) / \sigma_x^2\right) N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$- N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$= N_z\left(\frac{(-z + v T)}{\sigma_x \sqrt{T}}, \frac{(-x + v T)}{\sigma_y \sqrt{T}}, \rho, \right)$$

$$- \exp\left(2 v y / \sigma_y^2\right) N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$+ \exp\left(2 v x / \sigma_x^2\right) N_z\left(\frac{2 x - z + v T}{\sigma_x \sqrt{T}}, \frac{(-x + v T)}{\sigma_y \sqrt{T}}, \rho, \right)$$

$$- N_z\left(\frac{2 x - z + v T}{\sigma_x \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$- \exp\left(2 v (y-x) / \sigma_x^2\right) N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

$$- N_z\left(\frac{2 y - z + v T}{\sigma_y \sqrt{T}}, \frac{(-y + v T)}{\sigma_z \sqrt{T}}, \rho, \right)$$

where $N_z(\cdot, \cdot, \rho)$ refers as a bivariate standard normal distribution with instantaneous correlation coefficient $\rho = \rho_z = \sqrt{\tau/T}$ (or $\rho = -\sqrt{\tau/T}$). Using the probability law and formulas of double-
barrier-hitting probability, in-the-money probability of the double barrier puts has following explicit form
\[
Q\{\omega: M^S_\tau (\omega) \geq x \text{ or } m^S_\nu (\omega) \leq y\} \cap \{\omega: X(\omega, T) \leq z\}
\]
\[
= Q\{\omega: M^S_\tau (\omega) \geq x \text{ or } m^S_\nu (\omega) \leq y\} - Q\{\omega: M^S_\tau (\omega) \geq x \text{ or } m^S_\nu (\omega) \leq y\} \cap \{\omega: X(\omega, T) \geq z\};
\]
\[
Q\{\omega: M^S_\tau (\omega) \leq x \text{ and } m^S_\nu (\omega) \geq y\} \cap \{\omega: X(\omega, T) \leq z\}
\]
\[
= Q\{\omega: M^S_\tau (\omega) \leq x \text{ and } m^S_\nu (\omega) \geq y\} - Q\{\omega: M^S_\tau (\omega) \leq x \text{ and } m^S_\nu (\omega) \geq y\} \cap \{\omega: X(\omega, T) \geq z\};
\]
\[
Q\{\omega: M^S_\tau (\omega) \geq x \text{ and } m^S_\nu (\omega) \leq y\} \cap \{\omega: X(\omega, T) \leq z\}
\]
\[
= Q\{\omega: M^S_\tau (\omega) \geq x \text{ and } m^S_\nu (\omega) \leq y\} - Q\{\omega: M^S_\tau (\omega) \geq x \text{ and } m^S_\nu (\omega) \leq y\} \cap \{\omega: X(\omega, T) \geq z\}.
\]

Proposition 4. Given the available current information set involved with the dynamic of underlying asset price \(\mathcal{F}_0\), a double knock-in call contract with terminal payoff \([S(T) - K]_{\{M^S_\tau \geq H, m^S_\nu \leq L\}}^{-1}_{\{S(T) = K\}}\) has the present value
\[
C_{DKI}(0) = S(0) e^{-\delta T} \times \{N_2(c_1, c_2, \rho_\tau) + N_2(c_1, c_3, \rho_\nu) + [H/S(0)]^{2(\nu + \sigma^2_\tau)} \times [N_2(c_4, c_5, \rho_\nu) - N_2(c_7, c_8, \rho_\nu)]
\]
\[
- N_2(c_4, c_5, \rho_\nu) + [L/S(0)]^{2(\nu + \sigma^2_\tau)} \times [N_2(c_7, c_8, \rho_\nu) - N_2(c_9, c_10, \rho_\nu)]
\]
\[
- K e^{-\delta T} \times \{N_2(s_1, s_2, \rho_\nu) + N_2(s_1, s_3, \rho_\nu) + [H/S(0)]^{2\sigma^2_\tau} \times [N_2(s_4, s_5, \rho_\nu) - N_2(s_7, s_8, \rho_\nu)]
\]
\[
- N_2(s_4, s_5, \rho_\nu) + [L/S(0)]^{2\sigma^2_\tau} \times [N_2(s_7, s_8, \rho_\nu) - N_2(s_9, s_{10}, \rho_\nu)]
\]
where arguments to the bivariate standard normal distribution are
\[
c_1 = g(S(0)/K, \nu + \sigma^2_\tau, T);
\]
\[
s_2 = g(S(0)/H, \nu + \sigma^2_\tau, T);
\]
\[
c_3 = g(L/S(0), \nu + \sigma^2_\tau, T);
\]
\[
s_4 = g(H^2/[KS(0)], \nu + \sigma^2_\tau, T);
\]
\[
s_5 = g(S(0)/H, \nu, T);
\]
\[
c_6 = g((L/S(0))/H^2, -\nu - \sigma^2_\tau, T);
\]
\[
s_6 = g((L/S(0))/H, -\nu, T);
\]
\[
c_7 = g((L^2/[KS(0)], \nu + \sigma^2_\tau, T);
\]
\[
s_7 = g(L^2/[KS(0)], \nu + \sigma^2_\tau, T);
\]
\[
c_8 = g((L^2/[KS(0)], -\nu - \sigma^2_\tau, T);
\]
\[
s_8 = g((L^2/[HS(0)], -\nu - \sigma^2_\tau, T);
\]
\[
c_9 = g((L^2/[HS(0)], \nu + \sigma^2_\tau, T);
\]
\[
s_9 = g((L^2/[HS(0)], \nu + \sigma^2_\tau, T);
\]
\[
s_{10} = g((L^2/[HS(0)], \nu + \sigma^2_\tau, T);
\]
\[
c_{10} = g((L^2/[HS(0)], -\nu - \sigma^2_\tau, T);
\]
\[
s_{11} = g((L^2/[HS(0)], -\nu, T);
\]
\[
c_{11} = g((L^2/[HS(0)], -\nu, T);
\]
\[
s_{12} = g((L^2/[HS(0)], \nu, T);
\]
\[
c_{12} = g((L^2/[HS(0)], \nu, T);
\]
and the function has explicit form
\[
g(a, b, c)= [\ln(a) + bc]/(\sqrt{\sigma c}.)
\]
Proposition 4 contains several interesting special cases. For example, with letting \(\tau = 0\), the above formula could be degenerated as the famous Black-Scholes formula. Another case is that, if choosing \(H = S(0)\) or \(L = S(0)\), a double barrier call will be reduced as a single barrier call.

Now turn the attention to other types of the double barrier contract. A simple and efficient way for valuing these contracts is to straightforwardly make the application of Theorem 3 and of changing the measure. To provide more economical insight to the portfolio hedge and to the arbitrage trading, however, the doing we adopt here is to construct the in-out and put-call parities for double barrier option.

Recall the terminal payoff of double knock-in put
\[ P_{DKI}(T) = [K - S(T)] 1_{\{M^2_T \geq H | m^2_T \leq L\}} 1_{\{S(T) < K\}}. \]  

(3)

Since \( 1_{\{S(T) < K\}} = 1 - 1_{\{S(T) \geq K\}} \), we can rewrite the expression (3) as

\[ P_{DKI}(T) = [K - S(T)] (1 - 1_{\{S(T) \geq K\}}) 1_{\{M^2_T \geq H | m^2_T \leq L\}} = C_{DKI}(T) + [K - S(T)] 1_{\{M^2_T \geq H | m^2_T \leq L\}}. \]  

(4)

Thus the put-call parity of double knock-in option is yielded as (4). It is also known that the terms

\[ 1_{\{M^2_T \leq H | m^2_T \leq L\}} = 1 - 1_{\{M^2_T > H | m^2_T > L\}}, \]

\[ 1_{\{M^2_T > H | m^2_T > L\}} = 1_{\{M^2_T > H | m^2_T > L\}} - 1_{\{M^2_T > H | m^2_T > L\}}. \]

With the similar doing, other types of the double barrier contractual payoff become

\[ C_{DKO}(T) = [S(T) - K](1 - 1_{\{M^2_T \geq H | m^2_T \leq L\}}) 1_{\{S(T) \geq K\}} = C(T) - C_{DKI}(T); \]

\[ P_{DKO}(T) = [K - S(T)](1 - 1_{\{M^2_T \geq H | m^2_T \leq L\}}) 1_{\{S(T) \geq K\}} = P(T) - P_{DKI}(T); \]

\[ C_{UDDO}(T) = [S(T) - K](1_{\{M^2_T \leq H | m^2_T \geq L\}} - 1_{\{M^2_T > H | m^2_T > L\}}) 1_{\{S(T) \geq K\}} = C_{UO}(T) - C_{DKO}(T); \]

\[ P_{UDDO}(T) = [K - S(T)](1_{\{M^2_T \leq H | m^2_T \geq L\}} - 1_{\{M^2_T > H | m^2_T > L\}}) 1_{\{S(T) \geq K\}} = P_{UO}(T) - P_{DKO}(T); \]

\[ C_{DDO}(T) = [S(T) - K](1_{\{M^2_T \leq H | m^2_T \geq L\}} - 1_{\{M^2_T > H | m^2_T > L\}}) 1_{\{S(T) \geq K\}} = C_{DO}(T) - C_{DKO}(T); \]

\[ P_{DDO}(T) = [K - S(T)](1_{\{M^2_T \leq H | m^2_T \geq L\}} - 1_{\{M^2_T > H | m^2_T > L\}}) 1_{\{S(T) \geq K\}} = P_{DO}(T) - P_{DKO}(T); \]

where \( C() \) and \( P() \) denotes the standard call and put respectively; \( C_{UO}() \) and \( P_{UO}() \) denotes the up-and-out call and put respectively; \( C_{DO}() \) and \( P_{DO}() \) denotes the down-and-out call and put respectively. Main derivation of the in-out and put-call parities follows Proposition 2 and is also based on the martingale properties. Explicit formulas are displayed as below.

**Proposition 5.** Suppose that the reference economy allows for short selling, perfect asset liquidity, and self-financing investment activities. The information of observation to the economical uncertainty is public and available to all agents. For \( x = \ln[H/S(0)], \ y = \ln[L/S(0)], \) and \( \mu = \nu + \sigma^2 \), in-out and put-call parities of double barrier option are

\[ P_{DKI}(0) = C_{DKI}(0) - S(0) e^{-\delta T} N(-x - \mu \tau)/(\sigma_\delta \sqrt{\tau}) + N((y + \mu \tau)/(\sigma_\delta \sqrt{\tau})) + \exp(2 \mu y / \sigma_\delta^2) \]

\[ + Ke^{-\gamma T} Q((-x - \mu \tau)/(\sigma_\gamma \sqrt{\tau})) \]

\[ C_{DKO}(0) = C(0) - C_{DKI}(0); \]

\[ P_{DKO}(0) = P(0) - P_{DKI}(0); \]

\[ C_{UDDO}(0) = C_{UO}(0) - C_{DKO}(0); \]

\[ P_{UDDO}(0) = P_{UO}(0) - P_{DKO}(0); \]

\[ C_{DDO}(0) = C_{DO}(0) - C_{DKO}(0); \]

\[ P_{DDO}(0) = P_{DO}(0) - P_{DKO}(0); \]

where

\[ R(\omega: M^X_T(\omega) \geq x \text{ or } m^X_T(\omega) \leq y) = N((-x + \mu \tau)/(\sigma_\delta \sqrt{\tau})) + N((y + \mu \tau)/(\sigma_\delta \sqrt{\tau})) + \exp(2 \mu y / \sigma_\delta^2) \]

\[ \times [N((-y + \mu \tau)/(\sigma_\delta \sqrt{\tau})) - N((2y - x + \mu \tau)/(\sigma_\delta \sqrt{\tau}))]; \]

\[ + N((-x - \mu \tau)/(\sigma_\gamma \sqrt{\tau})) - N((-y - 2x - \mu \tau)/(\sigma_\gamma \sqrt{\tau}))]; \]

\[ - \exp(2 \mu x / \sigma_\gamma^2) - N((y - 2x + \mu \tau)/(\sigma_\gamma \sqrt{\tau}))]; \]

\[ - N((2y - 3x + \mu \tau)/(\sigma_\gamma \sqrt{\tau})) \exp(2 \mu (y - x) / \sigma_\delta^2); \]

\[ C(0) = S(0) e^{-\delta T} N(c_1) - Ke^{-\gamma T} N(s_1); \]

\[ P(0) = C(0) + Ke^{-\gamma T} - S(0) e^{-\delta T}; \]

\[ C_{UO}(0) = S(0) e^{-\delta T} [N_2(c_1, -c_2, \rho_\gamma) - [H/S(0)]^{2\mu / \sigma_\delta^2} N_2(c_4, c_5, \rho_\gamma)]; \]

\[ - Ke^{-\gamma T} [N_2(s_1, -s_2, \rho_\gamma) - [H/S(0)]^{2\mu / \sigma_\delta^2} N_2(s_4, s_5, \rho_\gamma)]; \]

\[ P_{UO}(0) = Ke^{-\gamma T} [N_2(-s_1, s_2, \rho_\gamma) - [H/S(0)]^{2\mu / \sigma_\delta^2} N_2(-s_4, s_5, \rho_\gamma)]; \]

\[ - S(0) e^{-\delta T} [N_2(-c_1, c_2, \rho_\gamma) - [H/S(0)]^{2\mu / \sigma_\delta^2} N_2(-c_4, c_5, \rho_\gamma)]; \]
Formulas given in Proposition 5 establish the linkages among the value of all types of double barrier options. These help investors not only detect whether arbitrage opportunities arise, but also learn how to replicate the double barrier contractual payoff in constructing their portfolio. We take the relation between double knock-in and knock-out calls as an example. It is known that, by in-out parity, $C_{DKO}(0) + C_{DKI}(0) = C(0)$, implying that the sum of the value of double knock-in and knock-out calls must equal to a standard call. If the trading prices escape from such a relation, market is no longer arbitrage-free. Arbitragers can earn the profits by shorting the over-valued options and longing the under-valued options simultaneously. Arbitrage trading has motivated a large number of studies. Since this is beyond the scope of our work, however, we do not attempt to put the focus on them. To compare with the literatures, some numerical analysis will be given in the next section.

3. Numerical Implementation

This section provides the numerical comparison between literature’s method and ours. For tractability in the analysis, assume that the trading contracts have the upper bound $H = 115$, the lower bound $L = 60$ the strike price $K = 100$, and the lifespan $T = 1$. Also let the underlying asset has zero-dividend payout ratio $\delta = 0$ and the annual volatility $\sigma_s = 0.2$, and the level of riskless interest rate is $r_d = 5\%$. Based on such a combination of parameters, Figures 1 and 2 plot the price of double knock-in and double knock-out put respectively for alternative choices of initial asset price. To have a closer inspection on the validity of our formula, these figures still offer the contrast with Monte Carlo simulation technique.
Figure 2: Double Knock-Out Put Prices as a Function of Initial Underlying Asset Prices

Firstly note that the double barrier put broadly becomes cheaper along with increasing initial underlying asset price. This is intuitive since a higher initial asset price results in a lower in-the-money likelihood for put. Besides, it is observable that the length of monitored period has asymmetric impact on the value of double knock-in and knock-out puts. If lengthening this period, the possibility that the contract turns into effective or nullified due to first-barrier-hitting will rise up. Hence the longer the monitored period, the higher the former prices but the lower the latter prices. Another interesting finding in the above figures is that the patterns of put price yielded by our close-form formula fit those from Monte Carlo simulation very well. Such a fact not only implies that the error in our model seems slight and moderate, but also verifies the validity of formulas from this paper.

Table 1: Double Knock-Out Call Prices

<table>
<thead>
<tr>
<th>T = 1/12</th>
<th>H</th>
<th>L</th>
<th>Price</th>
<th>Pelsser</th>
<th>Error ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ_s = 0.2</td>
<td>1500</td>
<td>500</td>
<td>25.1207</td>
<td>25.12</td>
<td>0.0028%</td>
</tr>
<tr>
<td></td>
<td>1200</td>
<td>800</td>
<td>24.7568</td>
<td>24.76</td>
<td>-0.0129%</td>
</tr>
<tr>
<td>σ_s = 0.3</td>
<td>1500</td>
<td>500</td>
<td>36.5842</td>
<td>36.58</td>
<td>0.0115%</td>
</tr>
<tr>
<td></td>
<td>1200</td>
<td>800</td>
<td>29.4473</td>
<td>29.45</td>
<td>-0.0092%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>T = 1/2</th>
<th>H</th>
<th>L</th>
<th>Price</th>
<th>Pelsser</th>
<th>Error ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ_s = 0.2</td>
<td>1500</td>
<td>500</td>
<td>66.1289</td>
<td>66.13</td>
<td>-0.0017%</td>
</tr>
<tr>
<td></td>
<td>1200</td>
<td>800</td>
<td>22.0812</td>
<td>22.08</td>
<td>-0.0054%</td>
</tr>
<tr>
<td>σ_s = 0.3</td>
<td>1500</td>
<td>500</td>
<td>67.8773</td>
<td>67.88</td>
<td>-0.0040%</td>
</tr>
<tr>
<td></td>
<td>1200</td>
<td>800</td>
<td>9.2649</td>
<td>9.26</td>
<td>0.0529%</td>
</tr>
</tbody>
</table>

Note: "Price" and "Pelsser" denote the call prices computed by the formulas from this paper and by the Laplace transform method of Pelsser (2000) respectively. Error ratio is calculated by (“Price” - “Pelsser”)/"Pelsser”.

To further clarify the benefits of our model, Table 1 presents a comparison for double knock-out call prices between the Laplace transform method (see Pelsser, 2000) and ours. The choices of several parameters for this table are borrowed from Pelsser’s article, including K=1000 and S(0)=1000. The numbers of error ratio are given in Column 6. No matter what the combination of parameters is, observe that the error ratios are quite slight and ranged from –0.1% to 0.1%. The sizes of error also make a response to the proceeding results that the accuracy of our model is stable. Since the formulas we derive are closed-form, the doing of this study seems more efficient and simpler than the
technique of Monte Carlo simulation and other methods that yield analytical solutions such as Laplace transform.

4. Conclusion
This paper makes a breakthrough in valuing the European zero-rebate contracts of double barrier option. To deviate from the existing models that yield an analytical or numerical solution solely (see Geman and Yor, 1996; Baldi et al., 1999; Pelsser, 2000; Schroder, 2000; Guillaume, 2003; and Lo and Hui, 2007), we derive the close-form solution with applying twice reflection principle. Our pricing formula is simpler and more efficient, and can be expressed as the nonlinear combinations of bivariate standard normal distribution. Behind the main formula, solutions of double barrier-hitting joint probability are also yielded. These play the useful tools in pricing the financial derivatives with path-dependent payoffs.

For further verifying the validity of our formula, this paper offers the numerical comparisons. The result seems exciting and indicates that the values computed by our formula fit those from Monte Carlo simulation and by Pelsser’s Laplace-transform approach very well. The sizes of error in the model are quite slight, which are ranged from –0.1% to 0.1%.

This paper can be applied in several further dimensions. The pricing of double barrier option with multi-underlying assets can be studied. More difficult extensions include the liquidity risk or market incompleteness. We expect that more applications will follow to devote to the studies of this subject.

References
Appendix
Proof of Theorem 2

Give any \( x \geq y \geq z \geq 0, \tau \leq T \), and \( X(t) = vt + \sigma_s W^\sigma(t) \) where \( v = r_d - \delta - 0.5 \sigma_s^2 \). For the convenience in our derivation, let \( x^\sigma \equiv x/\sigma_s \), \( y^\sigma \equiv y/\sigma_s \), \( z^\sigma \equiv z/\sigma_s \), and \( X^\sigma(t) \equiv X(t)/\sigma_s = \mu t + W^\sigma(t) \) where \( \mu \equiv v/\sigma_s \). Then we begin with a type of the double-barrier joint probability by choosing \( \tau = T \) and that can be expressed as, conditioned on the current information set \( F_0 \)

\[
Q(\omega: M^X_t(\omega) \geq x, m^X_t(\omega) \leq y, X(\omega, T) \leq z) = Q(M^x^\sigma \geq x^\sigma, m^x^\sigma \leq y^\sigma, X^\sigma(T) \leq z^\sigma) = E_Q(1_D|F_0)
\]

where \( D \equiv \{ \omega: M^x^\sigma(\omega) \geq x^\sigma, m^x^\sigma(\omega) \leq y^\sigma, X^\sigma(\omega, T) \leq z^\sigma \} \).

Allow \( P \) for the probability measure equivalent to \( Q \) such that its Radon-Nikodym derivative \( \xi(u), t \leq u \leq T \) is defined as

\[
\xi(u) := \frac{dP}{dQ}_{F_0}, \text{ } Q \text{-a.s.}
\]

with satisfying

\[
\xi(T) = \xi(0) - \int_{[0,T]} \xi(u) \mu dW^\sigma(u).
\]

Hence, the explicit solution of this Radon-Nikodym density process is

\[
dP/dQ = \exp[-\mu W^\sigma(T) - 0.5 \mu^2 T] = \exp[-\mu X^\sigma(T) + 0.5 \mu^2 T] = (dQ/dP)^{-1}.
\]

By Girsanov’s theorem, the process \( W^P(t), 0 \leq t \leq T \) under \( P \) defined by

\[
W^P(t) = W^Q(t) + \mu t = X^\sigma(t)
\]

follows a standard geometric Brownian motion.

Applying the reflection principle of Brownian motion \( X^\sigma(T) = 2x^\sigma - \hat{X}^\sigma(T) \) and the change of measure \( dP/dQ \), (A.1) becomes

\[
Q(M^x^\sigma \geq x^\sigma, m^x^\sigma \leq y^\sigma, X^\sigma(T) \leq z^\sigma) = E_P[(dQ/dP)1_D|F_0]
\]

\[
= E_P[\exp\{\mu X^\sigma(T) - 0.5 \mu^2 T\}1_D|F_0]
\]

\[
= \exp(2\mu x^\sigma)E_P[\exp\{-\mu \hat{X}^\sigma(T) - 0.5(-\mu)^2 T\}1_D|F_0]
\]

where

\[
\hat{D} \equiv \{ \omega: M^\hat{x}^\sigma(\omega) \geq x^\sigma, M^\hat{x}^\sigma(\omega) \geq 2x^\sigma - y^\sigma, \hat{X}^\sigma(\omega, T) \geq 2x^\sigma - z^\sigma \}.
\]

Since it’s known that \( x^\sigma \geq z^\sigma \) we get

\[
\{ \omega: M^\hat{x}^\sigma(\omega) \geq -x^\sigma \} \supset \{ \omega: \hat{X}^\sigma(\omega, T) \geq x^\sigma \} = \{ \omega: \hat{X}^\sigma(\omega, T) \geq 2x^\sigma - z^\sigma \}.
\]

Thus the underlying set can be reduced as

\[
\hat{D} \equiv \{ \omega: M^\hat{x}^\sigma(\omega) \geq 2x^\sigma - y^\sigma, \hat{X}^\sigma(\omega, T) \geq 2x^\sigma - z^\sigma \}.
\]

Again let \( \hat{P} \) be another martingale measure equivalent to \( P \) with corresponding Radon-Nikodym density process, for \( 0 \leq u \leq T \),

\[
h(u) := \frac{d\hat{P}}{dP}_{F_u}, \text{ } P \text{-a.s.}
\]

and

\[
h(T) = \exp[-\mu \hat{X}^\sigma(T) - 0.5(-\mu)^2 T]
\]

Using the technique of changing the measure with substituting (B.3) into (B.2) has

\[
Q(M^x^\sigma \geq x^\sigma, m^x^\sigma \leq y^\sigma, X^\sigma(T) \leq z^\sigma) = \exp(2\mu X^\sigma)\hat{P}(M^\hat{x}^\sigma \geq 2x^\sigma - y^\sigma, \hat{X}^\sigma(T) \geq 2x^\sigma - z^\sigma)
\]

Based on the probability law, (A.4) can be rewritten as
Firstly, it is known that (see Corollary B.3.4 in Musiela and Rutkowski, 1997)
\[ \hat{P}(M_t^{x'} \geq 2x^\sigma - y^\sigma, \hat{X}^\sigma(T) \leq 2x^\sigma - z^\sigma) = \exp(2\mu x^\sigma) \hat{P}(M_t^{x'} \geq 2x^\sigma - y^\sigma) - \hat{P}(M_t^{x'} \geq 2x^\sigma - y^\sigma, \hat{X}^\sigma(T) \leq 2x^\sigma - z^\sigma) \]

In deriving the other term in (A.4), we apply the similar technique to show that
\[ \hat{P}(M_t^{x'} \geq 2x^\sigma - y^\sigma, \hat{X}^\sigma(T) \leq 2x^\sigma - z^\sigma) = \exp(2\mu (y^\sigma - 2x^\sigma)) \hat{P}((\hat{X}^\sigma(T) \geq 2x^\sigma - 2y^\sigma + z^\sigma)\]
\[ = \exp(2\mu (y^\sigma - 2x^\sigma)) \mathcal{N}((2y^\sigma - 2x^\sigma - z^\sigma + \mu T)/\sqrt{T}) \]

where \( \hat{P} \) plays another martingale measure equivalent to \( \hat{P} \) that defines a specific process \( W^\hat{P}(t) = \hat{X}^\sigma(t) - \mu t = 4x^\sigma - 2y^\sigma - \hat{X}^\sigma(t) - \mu t \).

Putting (A.4.1) and (A.4.2) into (A.4) yields
\[ \mathcal{Q}(M_t^{x'} \geq x^\sigma, m_t^{x'} \leq y^\sigma, X^\sigma(T) \leq z^\sigma) = \exp(2\mu x^\sigma) \mathcal{N}(-2x^\sigma + y^\sigma - \mu T)/\sqrt{T}) \]
\[ + \exp(2\mu (y^\sigma - x^\sigma)) \mathcal{N}((-2x^\sigma + y^\sigma + \mu T)/\sqrt{T}) \]
\[ = \exp(2\mu (y^\sigma - x^\sigma)) \mathcal{N}((-2x^\sigma + y^\sigma - 2x^\sigma + z^\sigma + \mu T)/\sqrt{T}) \]

With letting \( x = z \), (A.5) becomes
\[ \mathcal{Q}(M_t^{x'} \geq x^\sigma, m_t^{x'} \leq y^\sigma, X^\sigma(T) \leq x^\sigma) = \exp(2\mu x^\sigma) \mathcal{N}((-2x^\sigma + y^\sigma - \mu T)/\sqrt{T}) \]
\[ + \exp(2\mu (y^\sigma - x^\sigma)) \mathcal{N}((-2x^\sigma + y^\sigma + \mu T)/\sqrt{T}) \]
\[ = \exp(2\mu (y^\sigma - x^\sigma)) \mathcal{N}((-2x^\sigma + y^\sigma + 2x^\sigma - z^\sigma + \mu T)/\sqrt{T}) \]

Since \( \{ \omega : M_t^{x'}(\omega) \geq x^\sigma \} \supset \{ \omega : X^\sigma(\omega, T) \geq x^\sigma \} \), we have
\[ \mathcal{Q}(M_t^{x'} \geq x^\sigma, m_t^{x'} \leq y^\sigma, X^\sigma(T) \geq x^\sigma) = \mathcal{Q}(m_t^{x'} \leq y^\sigma, X^\sigma(T) \geq x^\sigma) \]
\[ = \exp(2\mu y^\sigma) \mathcal{N}((-2x^\sigma + y^\sigma + 2x^\sigma - z^\sigma + \mu T)/\sqrt{T}) \]

In addition, by again using Corollary B.3.4 in Musiela and Rutkowski (1997), the formulas of single-barrier-type probabilities are yielded as
\[ \mathcal{Q}(M_t^{x'} \geq x^\sigma) = \mathcal{N}((-x + vT)/\sigma_s \sqrt{T}) + \mathcal{N}(2v x^\sigma) \mathcal{N}((-x - vT)/\sigma_s \sqrt{T}) \]
\[ \mathcal{Q}(m_t^{x'} \leq y^\sigma) = \mathcal{N}((y - vT)/\sigma_s \sqrt{T}) + \mathcal{N}(2v y^\sigma) \mathcal{N}((y + vT)/\sigma_s \sqrt{T}) \]

Then collecting (A.6)-(A.9) can obtain the formula of the double knock-in joint probability, that is,
\[ \mathcal{Q}(\omega : M_t^{x'}(\omega) \geq x \ or \ m_t^{x'}(\omega) \leq y) = \mathcal{Q}(M_t^{x'} \geq x) + \mathcal{Q}(m_t^{x'} \leq y) + \mathcal{Q}(M_t^{x'} \geq x, m_t^{x'} \leq y, X(T) \geq x) \]
\[ \mathcal{Q}(M_t^{x'} \geq x, m_t^{x'} \leq y, X(T) \leq x) \]

Given the following knock-in-out relation
Q(\omega: M_T^X(\omega) \geq x \text{ or } m_T^X(\omega) \leq y) = 1 - Q(\omega: M_T^X(\omega) \geq x \text{ and } m_T^X(\omega) \leq y)
= 1 - [Q(M_T^X \geq x, m_T^X \leq y, X(T) \geq x) + Q(M_T^X \geq x, m_T^X \leq y, X(T) \leq x)] ,
Q(\omega: M_T^X(\omega) \geq x \text{ and } m_T^X(\omega) \geq y) = Q(M_T^X \geq x) - [Q(M_T^X \geq x, m_T^X \leq y, X(T) \geq x) + Q(M_T^X \geq x, m_T^X \leq y, X(T) \leq x)] ,

and

Q(\omega: M_T^X(\omega) \leq x \text{ and } m_T^X(\omega) \leq y) = Q(m_T^X \leq y) - [Q(M_T^X \geq x, m_T^X \leq y, X(T) \geq x) + Q(M_T^X \geq x, m_T^X \leq y, X(T) \leq x)] ,

the formulas for other types of double barrier joint probability can be easily acquired also. This completes the proof of Theorem 2. Through the above technique, one could prove Theorem 3 and Propositions 4 and 5.